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Contractibility of Efficient Frontier of Simply Shaded Sets

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Abstract. In finite dimensional Euclidean space, we prove the contractibility of the efficient frontier of simply shaded sets. This work extends the result of Peleg [7], which confirms the contractibility of the efficient frontier in the convex case.

Key words: Simply shaded sets, Efficient frontier, Multiobjective problem, Contractibility

1. Introduction

In the present paper, we assume that the finite dimensional Euclidean space $\mathbb{R}^n$ is ordered by the positive cone $\mathbb{R}^n_+ := [0, +\infty[^n$. In this context, Peleg [7] states that the efficient frontier of a convex set of $\mathbb{R}^n$ is contractible. Our aim is to extend this result to the nonconvex sets. More precisely, we establish the contractibility of the efficient frontier for sets belonging to the class of simply shaded sets introduced by Benoist–Popovici in [2, 3]. The proof of this result is similar to Peleg's proof, the idea being to move in the topological interior of the considered set.

This work extends those of several authors [1, 4, 8] which confirm the connectedness of the efficient frontier for strictly quasiconcave vector maximization problems in particular cases.

In Section 2, we introduce the class of simply shaded sets and we motivate this concept by giving two important examples. In Section 3, we give basic properties of these sets. Under a compactness assumption, we establish in Section 4 the contractibility of the efficient frontier (see Theorem 4.2). Section 5 is devoted to an extension of Theorem 4.2 when the compactness assumption fails. Finally, an open problem is proposed in the conclusion.

2. Definitions of simply shaded sets and examples

Throughout this paper, we consider the n-dimensional Euclidean space $\mathbb{R}^n$ ($n \geqslant 2$) ordered by the positive cone $\mathbb{R}^n_+$. We denote by $\{e_i\}_{i=1}^n$ the canonical basis of $\mathbb{R}^n$ and it shall be convenient in the sequel to define e_i for each integer by putting $e_i = e_j$ if $j - i \in n\mathbb{Z}$. We also set $e = \sum_{i=1}^n e_i = (1, \ldots, 1)$. If $x \in \mathbb{R}^n$ and $\rho > 0$, $B(x, \rho) := \{z \in \mathbb{R}^n : \|z - x\| \leqslant \rho\}$ denotes the closed ball centered at x of radius ρ.

As usual, given a subset X in $\mathbb{R}^n$, we define the efficient frontier of X by

$$\mathrm{Eff}\, X := \{x \in X : X \cap (x + \mathbb{R}^n_+) = \{x\}\}.$$

It is well known that $\mathrm{Eff}\, X = \mathrm{Eff}\,(X - \mathbb{R}^n_+)$. Then, the study of the efficient frontier of X can be reduced to the study of the efficient frontier of a free disposal set, namely $Y := X - \mathbb{R}^n_+$. Recall that a subset Y of $\mathbb{R}^n$ is called free disposal in the sense of Debreu (see [5]) if $Y - \mathbb{R}^n_+ = Y$.

The aim of the article is to give assumptions on Y to get a strongly contractible efficient frontier. The efficient frontier of a free disposal set is not generally a connected set. Indeed, if $n := 2$ and $Y := \{(1, 0), (0, 1)\} - \mathbb{R}^2_+$, the efficient frontier of Y is reduced to $\{(1, 0), (0, 1)\}$. According to this example, the following assumption seems to be necessary to obtain good topological properties of the efficient frontier. This definition was previously introduced by Benoist–Popovici [2] in a more general framework. If x and z are two vectors of $\mathbb{R}^n$, we adopt the following convention in notations: $x \geqslant z$ means $x - z \in \mathbb{R}^n_+$, $x > z$ means $x \geqslant z$ and $x \neq z$, and $x \gg z$ means that $x - z \in \mathbb{R}^{\star n}_+$.

DEFINITION 2.1. A closed and free disposal set Y of $\mathbb{R}^n$ is said to be simply shaded if for any pair $(y, z) \in Y \times \mathrm{bd}\, Y$ we have

$$y \geqslant z \Longrightarrow y - \mathbb{R}_+(y - z) \subset \mathrm{bd}\, Y. \tag{1}$$

The following property shows that to prove that a set is simply shaded it suffices to verify condition (1) only when the vector $y - z$ belongs to an extremal ray of the cone $[0, +\infty[^n$.

PROPOSITION 2.2. *A closed and free disposal set Y of $\mathbb{R}^n$ is simply shaded if and only if, for any pair $(y, z) \in Y \times \mathrm{bd}\, Y$, for any integer $i \in \{1, \ldots, n\}$,*

$$y \in z + \mathbb{R}_+ e_i \implies y - \mathbb{R}_+(y - z) \subset \mathrm{bd}\, Y. \tag{2}$$

Proof. Suppose that (2) is true and let $(y, z) \in Y \times \mathrm{bd}\, Y$ such that $y \geqslant z$. Then, the vector $y - z$ has a representation $y - z = \sum_{i=1}^n \alpha_i e_i$ with $\alpha_i \geqslant 0$ for all i. From the free disposal assumption, to prove (1) it suffices to prove that $z - \mathbb{R}_+(y - z) \subset \mathrm{bd}\, Y$. Let $x \in z - \mathbb{R}_+(y - z)$. There exists $t \geqslant 0$ such that $x = z - t(y - z)$. Put $x_0 = z$ and $x_i = x_{i-1} - t\alpha_i e_i$ where $i \in \{1, \ldots, n\}$, and let us prove by induction that $x_i \in \mathrm{bd}\, Y$ for all $i \in \{0, \ldots, n\}$. The property is true for $i = 0$. Suppose that it is true for $i - 1$ $(1 \leqslant i \leqslant n)$, namely $x_{i-1} \in \mathrm{bd}\, Y$. Since $y \geqslant z + \alpha_i e_i \geqslant x_{i-1} + \alpha_i e_i$, we have $x_{i-1} + \alpha_i e_i \in Y$ from the free disposal assumption. By using (2) with the pair $(x_{i-1} + \alpha_i e_i, x_{i-1})$, we deduce $x_i \in \mathrm{bd}\, Y$, and so the property is true for i. Hence $x = x_n \in \mathrm{bd}\, Y$ and so (1) is proven. □

We give now two important classes of simply shaded sets.

EXAMPLE 2.3. A closed and free disposal convex set of $\mathbb{R}^n$ is simply shaded. Indeed, it is a direct consequence of the following property satisfied for any convex set Y: $\forall t \in]0, 1[$, $\forall (y, z) \in Y \times \mathrm{int}\, Y$, $ty + (1 - t)z \in \mathrm{int}\, Y$.

EXAMPLE 2.4. Let us consider the multiobjective problem

$$\begin{aligned}&\text{maximize} \quad g(c) = (g_1(c), \dots, g_n(c))\\ &c \in C\end{aligned}$$

where C is a nonempty compact convex subset of $\mathbb{R}^m$ and, for each integer $i \in \{1, \dots, n\}$, the function $g_i : C \to \mathbb{R}$ is continuous and strictly quasiconcave, i.e., for all $t \in]0, 1[$ and all $(c_1, c_2) \in C^2$ satisfying $g_i(c_1) \neq g_i(c_2)$ we have $g_i(tc_1 + (1-t)c_2) > \min(g_i(c_1), g_i(c_2))$. Then $Y = g(C) - \mathbb{R}^n_+$ is simply shaded.

Indeed, according to Proposition 2.2, it suffices to prove condition (2). Let $i \in \{1, \dots, n\}$ and let $(y_1, y_2) \in Y \times \mathrm{bd}\, Y$ such that $y_2 \in y_1 - \mathbb{R}^\star_+ e_i$. From the definition of Y, there exists $x_1 \in g(C)$ such that $x_1 \geqslant y_1$. Suppose, to the contrary, that the conclusion does not hold. There exists $y \in y_1 - \mathbb{R}_+ e_i$ such that $y \in \mathrm{int}\, Y$. There then exists $x \in Y$ such that $x \gg y$, and without any loss of generality we can suppose that $x \in g(C)$ from the definition of Y. There exists $(c_1, c) \in C^2$ such that $x_1 = g(c_1)$ and $x = g(c)$. For each integer k, set $z_k = g((1 - 1/(k+1))c_1 + 1/(k+1)c)$. For each integer $j \in \{1, \dots, n\}$ and for each integer $k \in \mathbb{N}$, we have

$$\begin{cases} \langle z_k, e_j \rangle > \min\,(\langle x_1, e_j \rangle, \langle x, e_j \rangle) & \text{if } \langle x_1, e_j \rangle \neq \langle x, e_j \rangle; \\ \langle z_k, e_j \rangle \geqslant \langle x, e_j \rangle & \text{if } \langle x_1, e_j \rangle = \langle x, e_j \rangle, \end{cases}$$

recalling that g_j is strictly quasiconcave.

Let $k \in \mathbb{N}$. Let $j \in \{1, \cdots, n\}$ such that $j \neq i$. If $\langle x_1, e_j \rangle \neq \langle x, e_j \rangle$, then $\langle z_k, e_j \rangle > \min(\langle x_1, e_j \rangle, \langle x, e_j \rangle) \geqslant \min(\langle y_1, e_j \rangle, \langle y, e_j \rangle) = \langle y_2, e_j \rangle$, that can be rewritten $\langle z_k, e_j \rangle > \langle y_2, e_j \rangle$. Otherwise, if $\langle x_1, e_j \rangle = \langle x, e_j \rangle$, then we have $\langle z_k, e_j \rangle \geqslant \langle x, e_j \rangle > \langle y, e_j \rangle = \langle y_2, e_j \rangle$, and we again can write $\langle z_k, e_j \rangle > \langle y_2, e_j \rangle$. On the other hand, for $j = i$ we have $\langle x_1, e_i \rangle \geqslant \langle y_1, e_i \rangle > \langle y_2, e_i \rangle$. Since the sequence $\{z_k\}$ converges to x_1, we can write $\langle z_k, e_i \rangle > \langle y_2, e_i \rangle$ for k large enough.

Thus, for k large enough $z_k \gg y_2$, which implies that $y_2 \in \mathrm{int}\, Y$ from the free disposal assumption. This contradicts the initial assumption on y_2, namely $y_2 \in \mathrm{bd}\, Y$.

3. Basic properties of simply shaded sets

In the sequel, we shall say that a subset X of $\mathbb{R}^n$ has compact sections if for each $x \in \mathbb{R}^n$ the section

$$X_+(x) := X \cap (x + \mathbb{R}^n_+)$$

is compact. It is easy to check that an arbitrary set with compact sections is closed. Moreover, if X is a free disposal set, X has compact sections if and only if for all $x \in X$ the section $X_+(x)$ is compact. Finally, if X is a compact subset of $\mathbb{R}^n$, the free disposal set $Y := X - \mathbb{R}^n_+$ has compact sections.

Let Y be a nonempty free disposal set with compact sections. For any vector $d > 0$ and any $y \in Y$, the subset $\{t \in \mathbb{R}_+ : y + td \in Y\}$ contains 0. Since the section $Y_+(y)$ is compact, there exists $\rho > 0$ such that $Y_+(y) \subset B(0, \rho)$. Then, for each t belonging to the previous subset we have $y + td \in B(0, \rho)$, which implies that $t \parallel d \parallel - \parallel y \parallel \leqslant \parallel y + td \parallel < \rho$. Hence the subset is also bounded from above by $(\parallel y \parallel + \rho)/\parallel d \parallel$. These remarks allow us to justify the following definition.

DEFINITION 3.1. For any vector $d > 0$, we can define a function λ_d from Y into $\mathbb{R}_+$ by the formula

$$\lambda_d(y) := \sup\{t \in \mathbb{R}_+ : y + td \in Y\}, \tag{3}$$

where $y \in Y$. Moreover, for each integer i we put

$$\lambda_i = \lambda_{e_i}.$$

We can easily check that $y + \lambda_d(y)d \in \operatorname{bd} Y$. Morerover, according to the free disposal assumption, $y + td \in Y$ if and only if $t \leqslant \lambda_d(y)$.

The following proposition gives various characterizations of simply shaded sets. Let us recall that the correspondance Y_+ is said to be lower semicontinuous (in short l.s.c.) at $y \in \operatorname{int} Y$ if for any $z \in Y_+(y)$ and any $\epsilon > 0$, there exists $\eta > 0$ such that $Y_+(y') \cap B(z, \epsilon) \neq \emptyset$ for all $y' \in B(y, \eta)$. We recall that Y_+ is l.s.c. on $\operatorname{int} Y$ if it is l.s.c. at every $y \in \operatorname{int} Y$.

PROPOSITION 3.2. *Let Y be a nonempty and free disposal set of $\mathbb{R}^n$ with compact sections. The following assertions are then equivalent.*
(A1) Y is simply shaded;
(A2) for each vector $d > 0$, the function λ_d is continuous on $\operatorname{int} Y$*;*
(A3) for each integer i, the function λ_i is continuous on $\operatorname{int} Y$*;*
(A4) the correspondance Y_+ is l.s.c. on $\operatorname{int} Y$.

Proof. $(A1) \implies (A2)$ Let $d > 0$ and $\{y_k\}$ be a sequence in $\operatorname{int} Y$ converging to an element $y \in \operatorname{int} Y$. We must show that the sequence $\{\lambda_d(y_k)\}$ converges to $\lambda_d(y)$. There exists $z \in Y$ such that $z \leqslant y_k$ for all integer k. Remarking that $z + \lambda_d(y_k)d \leqslant y_k + \lambda_d(y_k)d$ and $y_k + \lambda_d(y_k)d \in Y$, we also have from the free disposal assumption $z + \lambda_d(y_k)d \in Y$. Following the definition of $\lambda_d(z)$ we obtain $0 \leqslant \lambda_d(y_k) \leqslant \lambda_d(z)$ and the sequence $\{\lambda_d(y_k)\}$ is bounded. It then suffices to prove that $\lambda_d(y)$ is the unique point of accumulation of this sequence. Let $\{\lambda_d(y_{k_j})\}_j$ be a subsequence which converges to a nonnegative real t. Letting $j \to +\infty$ in the following relation $y_{k_j} + \lambda_d(y_{k_j})d \in \operatorname{bd} Y$ yields $y + td \in \operatorname{bd} Y$, and consequently $t \leqslant \lambda_d(y)$. Suppose that the inequality is strict. By applying (1) with the pair $(y + \lambda_d(y)d, y + td)$, we obtain the inclusion $y + \lambda_d(y)d - \mathbb{R}_+ d \subset \operatorname{bd} Y$ and in particular $y \in \operatorname{bd} Y$, which is impossible. Hence, the equality $t = \lambda_d(y)$ is satisfied.

$(A2) \implies (A4)$ Suppose that the correspondance Y_+ is not l.s.c. on $\operatorname{int} Y$. Then there exist $y \in \operatorname{int} Y$, $z \in Y_+(y)$, $\epsilon > 0$ and a sequence $\{y_k\}$ in $\operatorname{int} Y$ converging to

y such that for all integers k

$$Y_+(y_k) \cap B(z, \epsilon) = \emptyset. \tag{4}$$

Put $d = z - y$. Clearly we have $d > 0$. On one hand, for k large enough, $\| y - y_k \| \leqslant \epsilon/2$, which implies

$$\begin{aligned}\left\| z - \left(y_k + \left(1 - \frac{\epsilon}{2(\epsilon + \| d \|)} \right) d \right) \right\| &= \left\| y - y_k + \frac{\epsilon}{2(\epsilon + \| d \|)} d \right\| \\ &\leqslant \| y - y_k \| + \frac{\epsilon}{2} \\ &\leqslant \epsilon,\end{aligned}$$

or equivalently $y_k + (1 - \epsilon/2(\epsilon + \| d \|))\, d \in B(z, \epsilon)$. Then, from (4), we deduce that $y_k + (1 - \epsilon/2(\epsilon + \| d \|))\, d \notin Y$ and we obtain $\lambda_d(y_k) \leqslant 1 - \epsilon/2(\epsilon + \| d \|)$ following the definition of $\lambda_d(y_k)$. On the other hand, we have $\lambda_d(y) \geqslant 1$. These last two inequalities imply that the function λ_d is not continuous at y.

$(A4) \Longrightarrow (A3)$ Let i be an integer and $\{y_k\}$ be a sequence in $\operatorname{int} Y$ converging to an element y in $\operatorname{int} Y$. We must show that the sequence $\{\lambda_i(y_k)\}$ converge to $\lambda_i(y)$. As in the proof of implication (A1) $\Longrightarrow$ (A2), the sequence $\{\lambda_i(y_k)\}$ is bounded. Then it suffices to prove that $\lambda_i(y)$ is the unique point of accumulation of this sequence. Let $\{\lambda_i(y_{k_j})\}_j$ be a subsequence which converges to a nonnegative real t. Letting $j \to +\infty$ in the following relation $y_{k_j} + \lambda_i(y_{k_j})e_i \in \operatorname{bd} Y$ yields $y + te_i \in \operatorname{bd} Y$, and consequently $t \leqslant \lambda_i(y)$. Let us now prove the inverse inequality. Let $\epsilon > 0$. Since the correspondence Y_+ is l.s.c. at y, we have for j large enough $Y_+(y_{k_j}) \cap B(y + \lambda_i(y)e_i, \epsilon) \neq \emptyset$. Select an element z_j in $Y_+(y_{k_j}) \cap B(y + \lambda_i(y)e_i, \epsilon)$. Since $z_j \geqslant y_{k_j} + \langle z_j - y_{k_j}, e_i \rangle e_i$, we deduce $y_{k_j} + \langle z_j - y_{k_j}, e_i \rangle e_i \in Y$ from the free disposal assumption. Then, following the definition of $\lambda_i(y_{k_j})$, $\lambda_i(y_{k_j}) \geqslant \langle z_j - y_{k_j}, e_i \rangle$. Letting the inferior limit as $j \to +\infty$ in this last inequality yields $t \geqslant \lambda_i(y) - \epsilon$. Finally, taking $\epsilon \to 0$, we get $t \geqslant \lambda_i(y)$. Hence we conclude $\lambda_i(y) = t$.

$(A3) \Longrightarrow (A1)$ Suppose that Y is not simply shaded. According to Proposition 2.2, there exist $y \in Y$, $\alpha > 0$ and an integer i such that $y - \alpha e_i \in \operatorname{bd} Y$ and $y - \mathbb{R}_+ e_i \not\subset \operatorname{bd} Y$. Then, $y - \beta e_i \in \operatorname{int} Y$ for some $\beta > \alpha$, or equivalently there exists $\epsilon > 0$ such that $B(y - \beta e_i, \sqrt{n}\epsilon) \subset Y$. Then the sequence $\{z_k\}$ defined for each integer k by $z_k := (y - \beta e_i) + \epsilon/(k+2)\, e$ is included in $\operatorname{int} Y$ and converges to $y - \beta e_i$. We shall prove now that the sequence $\{\lambda_i(z_k)\}$ does not converge to $\lambda_i(y - \beta e_i)$. On the one hand, remarking that $z_k + (\beta - \alpha)e_i \gg y - \alpha e_i$ and recalling that $y - \alpha e_i \in \operatorname{bd} Y$, we have $z_k + (\beta - \alpha)e_i \notin Y$. Following the definition of $\lambda_i(z_k)$, we get $\lambda_i(z_k) \leqslant \beta - \alpha$. On the other hand, by definition of $\lambda_i(y - \beta e_i)$, we have $\lambda_i(y - \beta e_i) \geqslant \beta$. These two last inequalities allow us to conclude. $\square$

REMARK 3.3. The assertion (A4) given in Proposition 3.2 has been used by Peleg [7, Lemma 4.5] to prove the contractibility of the efficient set of a convex subset. Here we shall use the characterization (A3).

The following example shows that the functions λ_i associated to a simply shaded set Y are not necessarily continuous on $\operatorname{bd} Y$. However, from the characterization (A3), they are continuous on $\operatorname{int} Y$.

EXAMPLE 3.4. If $n := 3$, let X be the convex hull of $B((0, 0, 0), 1) \cup \{(1, 0, 1)\}$ and let Y be the convex set $X - \mathbb{R}^3_+$. The sequence $\{y_k\} := \{(\cos 1/k, \sin 1/k, 0)\}$ converges to $(1, 0, 0)$, but $\{\lambda_3(y_k)\}$ is the zero sequence, which does not converge to $\lambda_3(1, 0, 0) = 1$. Hence λ_3 is not continuous at the boundary point $(1, 0, 0)$. Also remark that the correspondence Y_+ is not l.s.c. at this point.

Finally, we give a property of simply shaded sets which play a key role to obtain the contractibility of the efficient frontier. This property is given in [2] in a more general framework. Its proof is a straighforward consequence of (1).

PROPOSITION 3.5. *Let Y be a simply shaded set of $\mathbb{R}^n$ with compact sections, let y be an element of* $\operatorname{int} Y$ *and let i be an integer. Then, for each $t < \lambda_i(y)$, we have $y + te_i \in \operatorname{int} Y$.*

4. The main results

We remember that a nonempty subset A of $\mathbb{R}^n$ is said to be strongly contractible relatively to a point a in A if there exists a continuous map h_A from $[0, 1] \times A$ into A such that $h_A(0, .)$ is the identity function on A, $h_A(1, .)$ is the constant function equal to a, and $h_A(t, a) = a$ for all $t \in [0, 1]$. To prove the strong contractibility of a set, it is often useful to apply the following classical result.

PROPOSITION 4.1. *Let A and B be two subsets of $\mathbb{R}^n$ such that $B \subset A$. We suppose that B is a retract of A, i.e. that there exists a continuous map f from A into B such that $f(b) = b$ for all $b \in B$ (we shall say that f is a retraction from A into B). If A is strongly contractible relatively to a point b of B, then B is also strongly contractible relatively to the same point b.*

Proof. We just need to consider the composed map $h_B = f \circ \tilde{h}_A$, where $\tilde{h}_A$ denotes the restriction of h_A to the subset $[0, 1] \times B$. □

Peleg [7] has shown that any nonempty closed, convex subset of $\mathbb{R}^n$ with compact sections has a contractible efficient frontier. The following theorem generalizes this property for simply shaded sets. Remark that Benoist–Popovici [3] have obtained recently this result in the three-dimensional Euclidean space.

THEOREM 4.2. *Let Y be a nonempty simply shaded set of $\mathbb{R}^n$ with compact sections. Then its efficient frontier is strongly contractible.*

Proof. Put

$$A := \operatorname{Eff} Y \cup \operatorname{int} Y \quad \text{and} \quad B := \operatorname{Eff} Y.$$

In step 1, we shall build a function f from A into Y and we shall show in step 2 that this function is a retraction from A into B. In step 3, we shall prove that A is strongly contractible relatively to each of its points and in particular to a given point of B (according to step 2, $B \neq \emptyset$). Then, according to Proposition 4.1, we can conclude that B is strongly contractible.

Step 1. We build a function f from A into Y.

We can define for each i a function $r_i : A \to Y$ by $r_i(a) := a + (\lambda_i(a)/2)e_i$ for all $a \in A$. Indeed, the relation $a + \lambda_i(a)e_i \in Y$ implies from the free disposal assumption that $r_i(a) \in Y$, recalling that $\lambda_i(a) \geqslant 0$. The properties of the function r_i are given in the following lemma.

LEMMA 4.3. *Let i be an integer. Then the following holds*

(i) *for all $a \in A$, $r_i(a) \geqslant a$;*
(ii) *for all $b \in B$, $r_i(b) = b$;*
(iii) $r_i(A) \subset A$*;*
(iv) *the function r_i is continuous.*

Proof. (i) and (ii) are clear.
(iii) Two eventualities can arise from the definition of A. Either $a \in \operatorname{int} Y$, hence $r_i(a) \in \operatorname{int} Y \subset A$ from Proposition 3.5. Or $a \in \operatorname{Eff} Y$, hence in this case $r_i(a) = a \in \operatorname{Eff} Y \subset A$.
(iv) Let $a \in A$ and let us prove that r_i is continuous at a. It is a consequence of Proposition 3.2 (and the fact that $\operatorname{int} Y$ is an open set) if $a \in \operatorname{int} Y$. We can now suppose that $a \in \operatorname{Eff} Y$. Let $\{a_k\}$ be a sequence in A converging to a. From (i), we have $r_i(a_k) \geqslant a_k$ for each integer k. Then, since the sections are compact, the sequence $\{r_i(a_k)\}$ is bounded and it suffices now to show that $r_i(a)$ is the unique point of accumulation of this sequence. Let $\{r_i(a_{k_j})\}_j$ be a subsequence which converges to a point $b \in Y$. Letting $j \to +\infty$ in the relation $r_i(a_{k_j}) \geqslant a_{k_j}$ yields $b \geqslant a$. We conclude that $b = a$ recalling that a is an efficient point of Y. □

Now, according to Lemma 4.3, the function $f_k := r_k \circ \cdots \circ r_0$ from A into Y is well-defined and continuous. Moreover for $a \in A$ the sequence $\{f_k(a)\}$ is increasing, i.e. $\ldots \geqslant f_k(a) \geqslant \ldots \geqslant f_1(a) \geqslant f_0(a)$, and is bounded from above since the section $Y_+(a)$ is compact. Hence the sequence $\{f_k\}$ converges pointwise to a function $f : A \to Y$.

Step 2. We prove in this step that f is a retraction from A to B

Firstly let us prove that $f(A) \subset B$. Suppose, to the contrary, that there exists $y \in A$ such that $f(y) \notin \operatorname{Eff} Y = B$. There then exists $z \in Y$ such that $z > f(y)$. From the free disposal assumption, we can suppose that z can be written $z = f(y) + te_i$, for some $i \in \{1, \dots, n\}$ and some $t > 0$. Since the sequence $\{f_k(y)\}$ converges to $f(y)$, we have for k large enough

$$f_k(y) \gg f(y) - te. \tag{5}$$

Select such an integer with the additional condition $(k+1)-i \in n\mathbb{Z}$, which assures that $f_{k+1} = r_i \circ f_k$. We have $f(y) \geqslant f_k(y) + \langle f(y) - f_k(y), e_i\rangle e_i$ recalling that $f(y) \geqslant f_k(y)$ and $z \geqslant f_k(y) + (\langle f(y) - f_k(y), e_i\rangle + t)\, e_i$ recalling the equality $z = f(y)+te_i$. Hence, following the definition of $\lambda_i(f_k(y))$, we obtain $\lambda_i(f_k(y)) \geqslant \langle f(y) - f_k(y), e_i\rangle + t$, which implies

$$f_{k+1}(y) = f_k(y) + \frac{\lambda_i(f_k(y))}{2} e_i \geqslant f_k(y) + \frac{1}{2}\left(\langle f(y) - f_k(y), e_i\rangle + t\right) e_i. \tag{6}$$

Thus, looking at coordinate i of (5) and (6), we obtain

$$\langle f_k(y), e_i\rangle > \langle f(y), e_i\rangle - t \quad \text{and} \quad \langle f_{k+1}(y), e_i\rangle \geqslant \frac{1}{2}\langle f_k(y) + f(y), e_i\rangle + \frac{t}{2}.$$

Combining these last inequalities, we conclude $\langle f_{k+1}(y), e_i\rangle > \langle f(y), e_i\rangle$, which contradicts the fact that $f(y) \geqslant f_{k+1}(y)$. This proves that $f(A) \subset B$.

Moreover, from Lemma 4.3, $f_k(b) = b$ for all $b \in B$ and for all integers k. Letting $k \to +\infty$ in this last equality yields $f(b) = b$.

To conclude the step, it suffices now to prove that the function f is continuous. Let $\{y_k\}$ be a sequence in A converging to a point y of A. We must prove that the sequence $\{f(y_k)\}$ converges to $f(y)$. Given $\epsilon > 0$, we assert that there exists $\eta > 0$ such that

$$f(A) \cap f(y) + [-\eta, +\infty[^n \subset B(f(y), \epsilon). \tag{7}$$

Indeed, otherwise we can find a sequence $\{z_k\}$ in $f(A)$ such that for all k

$$z_k \geqslant f(y) - \frac{1}{k+1} e \quad \text{and} \quad z_k \notin B(f(y), \epsilon). \tag{8}$$

Since Y has compact sections and since $f(A) \subset Y$, the sequence $\{z_k\}$ is bounded. Extracting subsequence if necessary, we may assume that this sequence converges to an element z of Y. Letting $k \to +\infty$ in (8) yields $z \geqslant f(y)$ and $z \notin B(f(y), \epsilon/2)$. It means that $f(y) \notin \operatorname{Eff} Y$, which contradicts the inclusion $f(A) \subset B$. Hence (7) is proven.

Now, since the sequence $\{f_k(y)\}$ converges to $f(y)$, there exists an integer k_0 such that $f_{k_0}(y) \geqslant f(y) - \eta/2e$. By continuity of function f_{k_0} and since $f \geqslant f_{k_0}$, we obtain $f(y_k) \geqslant f_{k_0}(y_k) \geqslant f(y) - \eta e$ for k large enough. According to (7), we conclude that $f(y_k) \in B(f(y), \epsilon)$ for k large enough.

Step 3. We prove in this step that A is strongly contractible relatively to each of its points.

Let y_0 be a point of A. Fix $\rho > 0$ such that $B(e,\rho) \subset \mathbb{R}_+^{\star n}$. Put $K := \bigcup_{t \geqslant 0} tB(e,\rho)$ which is a closed convex cone with nonempty interior satisfying $K \setminus \{0\} \subset]0,+\infty[^n$. Since $y_0 -]0,+\infty[^n$ is an open set included in Y, it is also included in $\operatorname{int} Y$. Consequently $y_0 - (K \setminus \{0\}) \subset \operatorname{int} Y \subset A$, which implies

$$y_0 - K \subset A \tag{9}$$

Firstly let us prove that the function k from $\mathbb{R}^n$ into $\mathbb{R}$ defined for each $y \in \mathbb{R}^n$ by $k(y) := \sup\{t \in \mathbb{R} : y + te \in y_0 - K\}$ is well-defined. Let $y \in \mathbb{R}^n$. If $y \neq y_0$ the equality $y - \| y_0 - y \|/\rho e = y_0 - \| y_0 - y \|/\rho\,(e + \rho y_0 - y/\| y_0 - y \|)$ implies that $-\| y_0 - y \|/\rho$ belongs to the set $\{t \in \mathbb{R} : y + te \in y_0 - K\}$, and consequently it is nonempty. This property is also true if $y = y_0$. Moreover, if t belongs to the previous set, we have $y_0 - y \geqslant te$, which implies that $\langle y_0 - y, e_1\rangle \geqslant t$ and so this subset is bounded from above. Hence $k(y)$ is well-defined. Moreover, since $y_0 - K$ is closed, we easily check that $y + k(y)e \in y_0 - K$. Now let us prove that k is a continuous function. Let $(y_1, y_2) \in \mathbb{R}^{2n}$. If $y_1 \neq y_2$, the equality

$$\begin{aligned} &y_2 + (k(y_1) - \| y_2 - y_1 \|/\rho)\, e = \\ &\qquad y_1 + k(y_1)e - \| y_2 - y_1 \|/\rho\,(e - \rho y_2 - y_1/\| y_2 - y_1 \|) \end{aligned}$$

implies that $k(y_1) - \| y_2 - y_1 \|/\rho \leqslant k(y_2)$, or equivalently $k(y_1) - k(y_2) \leqslant \| y_2 - y_1 \|/\rho$. This last inequality is also true if $y_1 = y_2$. Since y_1 and y_2 play the same role, we conclude that $| k(y_2) - k(y_1) | \leqslant \| y_2 - y_1 \|/\rho$ and k is continuous.

It is now easy to deduce that the function h_A from $[0,1] \times A$ into $\mathbb{R}^n$ defined for each $(t, y) \in [0,1] \times A$ by

$$h_A(t,y) := \begin{cases} y + 2tk(y)e & \text{if } 0 \leqslant t \leqslant \frac{1}{2} \\ 2(1-t)(y + k(y)e) + (2t-1)y_0 & \text{if } \frac{1}{2} \leqslant t \leqslant 1 \end{cases}$$

is also continuous. Moreover, it is obvious that $h_A(0,.)$ is the identity function, that $h_A(1,.)$ is the constant function y_0, and that $h_A(t, y_0) = y_0$ for all $t \in [0,1]$. To conclude, it suffices now to prove the inclusion $h_A([0,1] \times A) \subset A$. Let $(t,y) \in [0,1] \times A$. If $1/2 \leqslant t \leqslant 1$, $h_A(t,y)$ belongs to the line segment joining $y + k(y)e$ and y_0. These last two points belong to the convex set $y_0 - K$. Hence $h_A(t,y) \in y_0 - K$, and so $h_A(t,y) \in A$ from (9). The case $t = 0$ is evident since $h_A(t,y) = y$. Suppose now that $0 < t \leqslant 1/2$. If $k(y) < 0$, then $y \gg h_A(t,y)$ and so $h_A(t,y) \in \operatorname{int} Y \subset A$ from the free disposal assumption. If now $k(y) \geqslant 0$, the equality $h_A(t,y) = y + k(y)e + (2t-1)k(y)e$ implies that $h_A(t,y) \in (y_0 - K) - K = y_0 - K$ and again we have $h_A(t,y) \in A$ from (9). □

Applying Theorem 4.2 for convex sets (see Example 2.3), we retrieve Peleg's result.

COROLLARY 4.4. *Let X be a nonempty closed convex set of $\mathbb{R}^n$ with compact sections. Then its efficient frontier is contractible.*

Proof. Put $Y := X - \mathbb{R}^n_+$ and recall that $\operatorname{Eff} Y = \operatorname{Eff} X$. Y is a closed, free disposal and nonempty convex set with compact sections. Then, according to Example 2.3, Y is a nonempty simply shaded set with compact sections and, applying Theorem 4.2, we conclude that $\operatorname{Eff} X$ is contractible. □

We can also state a result of contractibility for the efficient frontier of a multiobjective problem (see Example 2.4).

COROLLARY 4.5. *Under the assumptions of Example 2.4, the efficient frontier of the multiobjective problem, i.e., the efficient frontier of $g(C)$, is strongly contractible.*

Proof. Put $Y = g(C) - \mathbb{R}^n_+$ and recall that $\operatorname{Eff} g(C) = \operatorname{Eff} Y$. Y is a closed, free disposal and nonempty set with compact sections. Moreover, according to Example 2.4, Y is nonempty and simply shaded. Hence, applying Theorem 4.2, we conclude that $\operatorname{Eff} g(C)$ is strongly contractible. □

In the convex case, when $\operatorname{Eff} Y$ is closed, Peleg has proved that it is a retract of Y. In the case of simply shaded sets, we can now easily obtain a similar result as shown the following theorem.

THEOREM 4.6. *Let Y be a nonempty simply shaded set of $\mathbb{R}^n$ with compact sections. If its efficient frontier is closed, then it is a retract of Y.*

Proof. For $y \in Y$, let $d(y, \operatorname{Eff} Y)$ be the distance between y and $\operatorname{Eff} Y$. Using the notations of the proof of Theorem 4.2, we easily check that the function $y \mapsto f(y - d(y, \operatorname{Eff} Y)e)$ from Y into $\operatorname{Eff} Y$ is a retraction. □

5. An extension of Theorem 4.1 without any compactness assumption

Throughout this section, we suppose that Y is a proper (i.e., $Y \neq \emptyset$ and $Y \neq \mathbb{R}^n$) simply shaded set of $\mathbb{R}^n$. The aim of this section is to suppress the compactness assumption in Theorem 4.2. In the sequel, we shall replace it by the following one:

$$\text{for all } y \in Y \text{ and for all } i \in \mathbb{N},\ y + \mathbb{R}e_i \not\subset \operatorname{bd} Y. \tag{10}$$

In others words, assumption (10) means that no affine line generated by an e_i is included in the boundary of Y. Clearly, if Y has compact sections then Y satisfies assumption (10). The following proposition shows that the converse is true if Y is a convex set.

PROPOSITION 5.1. *Let Y be a proper closed and free disposal convex subset of $\mathbb{R}^n$. If no affine line generated by an e_i is included in the boundary of Y then Y has compact sections.*

Proof. We prove the contrapositive assertion. Suppose that Y has a section $Y_+(y)$ which is not compact for some $y \in Y$. There then exists an unbounded sequence $\{y_k\}$ included in $Y_+(y)$. Subsequencing if necessary, we may suppose from the free disposal assumption that there exists $i \in \{1, \ldots, n\}$ such that, for all integer k, $y_k \in y + \mathbb{R}_+ e_i$. Consequently $y + \mathbb{R}e_i \subset Y$.

If $\partial Y = \emptyset$ then $Y = \operatorname{int} Y$ is both an open and a closed subset of $\mathbb{R}^n$ which contradicts that Y is proper. We can then choose an element z in ∂Y. Since Y is a closed convex set, the element $z + te_i = \lim_{k \to +\infty} (1 - (1/k))z + 1/k(y + kte_i)$ belongs to Y for all $t \in \mathbb{R}$. Hence $z + \mathbb{R}e_i \subset Y$ which implies that $z + \mathbb{R}e_i \subset \partial Y$ recalling that $z \in \partial Y$ and Y is a simply shaded set (see Example 2.3.). □

In fact, to assume that (10) is true is a weak assumption. Indeed, for $n := 2$, the sets Y which do not satisfy assumption (10) are only the half-spaces $]-\infty, \alpha] \times \mathbb{R}$ and $\mathbb{R} \times]-\infty, \alpha]$ with $\alpha \in \mathbb{R}$. In particular the set defined by $Y := \{(x, y) \in]0, +\infty[^2 : y = 1/x\} - [0, +\infty[^2$ does not have the compact sections property, but it verifies assumption (10).

The following example shows that assumption (10) is necessary to guarantee the connectedness of the efficient frontier.

EXAMPLE 5.2. If $n := 3$, let $Y := \{(x, y, z) \in \mathbb{R}^3 : y + (1 + z_+)x \leqslant 0$ and $x + (1 + z_+)y \leqslant 0\}$ where we set $z_+ := \max(z, 0)$. If we assume $z > 0$, the set $\{(x, y) \in \mathbb{R}^2 : (x, y, z) \in Y\}$ is the intersection of the two distinct half-planes defined by the inequalities $y + (1 + z)x \leqslant 0$ and $x + (1 + z)y \leqslant 0$. Otherwise, it is the half-space defined by the equality $x + y \leqslant 0$. By sketching these sets, it is easy to see that Y is simply shaded. But $\operatorname{Eff} Y = (\operatorname{bd} Y \setminus \mathbb{R}e_3) \cap (\mathbb{R}^2 \times \mathbb{R}_+)$ and consequently the efficient frontier of Y is not a connected set. Remark that assumption (10) is not satisfied in this example since the line $\mathbb{R}e_3$ is included in the boundary of Y.

The following theorem extends Theorem 4.2 by showing that assumption (10) is sufficient to guarantee the connectedness of the efficient frontier of Y.

THEOREM 5.3. *Let Y be a proper simply shaded set of $\mathbb{R}^n$ with no affine line generated by an e_i included in the boundary of Y. Its efficient frontier is then strongly contractible.*

Proof. As in the proof of Theorem 4.2, we shall build a retraction from $A = \operatorname{Eff} Y \cup \operatorname{int} Y$ to $B = \operatorname{Eff} Y$. The end result of the proof will be the same.

Although the functions λ_i are not generally defined in all the set Y, we can consider their restrictions to the subset $O := \{y \in Y : Y_+(y)$ is bounded$\}$. More precisely, for each integer i, we can define a function $\tilde{\lambda}_i$ from O into $\mathbb{R}_+$ by the formula $\tilde{\lambda}_i(y) := \sup\{t \in \mathbb{R}_+ : y + te_i \in Y\}$ for all $y \in O$. With these definitions, we have the following lemma.

LEMMA 5.4. *Let i be an integer. The following then holds:*

(i) $\operatorname{bd} Y \subset O$;

(ii) *O is an open set for the induced topology on Y;*

(iii) *the function $\tilde{\lambda}_i$ is locally bounded, i.e. for all $y \in O$ there exist $\epsilon > 0$ and $M > 0$ such that $\tilde{\lambda}_i(z) \leqslant M$ for all $z \in O \cap B(y, \epsilon)$;*

(iv) *the function $\tilde{\lambda}_i$ is continuous on $O \cap \operatorname{int} Y$.*

Proof. Firstly, let us prove the relation $Y \setminus O = \{y \in Y : y + \mathbb{R}e_i \subset Y$ for some $i \in \{1, \ldots, n\}\}$. Indeed, if $y \in Y \setminus O$, we can find a sequence $\{y_k\}$ in $Y_+(y)$ with $\lim_{k \to +\infty} \| y_k \| = +\infty$. There then exists an integer $i \in \{1, \ldots, n\}$ such that $\lim_{k \to +\infty} \langle y_k, e_i \rangle = +\infty$. Let k be an integer. Since $y_k \geqslant y + \langle y_k - y, e_i \rangle e_i$ and $y_k \in Y$, we have $y + \langle y_k - y, e_i \rangle e_i \in Y$ from the free disposal assumption. Letting $k \to +\infty$ in this last relation yields $y + \mathbb{R}e_i \subset Y$. The converse inclusion is immediate.
(i) Let $y \in \operatorname{bd} Y$. Suppose, to the contrary, that $y \notin O$. There exists an integer i such that $y + \mathbb{R}e_i \subset Y$. Applying (2) with the pairs $(y + (k+1)e_i, y)$ where k is an arbitrary integer, we get $y +]-\infty, k+1]e_i \subset \operatorname{bd} Y$. Then, passing to the limit as $k \to +\infty$ yields $y + \mathbb{R}e_i \subset \operatorname{bd} Y$, which contradicts assumption (10).
(ii) Let us prove that $Y \setminus O$ is closed for the induced topology on Y. Let $\{y_k\}$ be a sequence in $Y \setminus O$ converging to an element y of Y. For each integer k there exists an integer i_k in $\{1, \ldots, n\}$ such that $y_k + \mathbb{R}e_{i_k} \subset Y$. Subsequencing if necessary, we may suppose that the index i_k does not depend upon k. Since Y is closed, letting $k \to +\infty$ in the last inclusion yields $y + \mathbb{R}e_i \subset Y$ for some integer i in $\{1, \ldots, n\}$. Thus, we conclude that $y \in Y \setminus O$.
(iii) Let $y \in O$. From (ii), there exists $\epsilon > 0$ such that $Y \cap B(y, \sqrt{n}\epsilon) \subset O$. In particular $y - \epsilon e \in O$, which means that the set $Y_+(y - \epsilon e)$ is bounded. Remarking that this bounded set is a neighbourhood of y, we easily conclude.
(iv) It suffices to again take the proof of the implication $(A1) \Longrightarrow (A2)$ of Proposition 3.2, by using (iii) to prove that $\{\tilde{\lambda}_i(y_k)\}$ is bounded. □

We can now define for each integer i a function $\tilde{r}_i$ from $O \cap A$ into Y such that $\tilde{r}_i(a) := a + (\tilde{\lambda}_i(a)/2)e_i$ for all $a \in O \cap A$. The properties of the function $\tilde{r}_i$ are given in the following lemma. Its proof is similar to the proof of Lemma 4.3 (iii) (some justifications can be given by using Lemma 5.4).

LEMMA 5.5. *Let i be an integer. The following then holds:*

(i) *for all $a \in O \cap A$, $\tilde{r}_i(a) \geqslant a$;*

(ii) *for all $b \in B$, $\tilde{r}_i(b) = b$;*

(iii) *$\tilde{r}_i(O \cap A) \subset O \cap A$;*

(iv) *the function $\tilde{r}_i$ is continuous.*

We must now construct a continuous function from A into $A \cap O$. To continue, we need the following lemma.

LEMMA 5.6. *It is possible to define the function λ_e from Y into $\mathbb{R}_+$ by the formula (3) with $d = e$. Moreover, λ_e is continuous.*

Proof. Firstly let us prove that λ_e is well-defined. Let $y \in Y$. The subset $\{t \in \mathbb{R}_+ : y + te \in Y\}$ contains 0. Moreover, since $Y \neq \mathbb{R}^n$, there exists a vector $z \notin Y$. Then, $y + te \geqslant z$ for t large enough, and consequently $y + te \notin Y$ from the free disposal assumption. Hence the previous set is also bounded and $\lambda_e(y)$ is well-defined. Since Y is closed, we easily check that $y + \lambda_e(y)e \in Y$.

Now let us prove that λ_e is a continuous function. Let $(y_1, y_2) \in Y^2$. The vector $y_2 + (\lambda_e(y_1) - \| y_2 - y_1 \|)e = y_1 + \lambda_e(y_1)e + (y_2 - y_1 - \| y_2 - y_1 \| e)$ belongs to Y from the free disposal assumption. Following the definition of $\lambda_e(y_2)$, we deduce that $\lambda_e(y_1) - \| y_2 - y_1 \| \leqslant \lambda_e(y_2)$, which is equivalent to $\lambda_e(y_1) - \lambda_e(y_2) \leqslant \| y_2 - y_1 \|$. Since y_1 and y_2 play the same role, we conclude that $| \lambda_e(y_1) - \lambda_e(y_2) | \leqslant \| y_2 - y_1 \|$, and so λ_e is continuous. □

Thanks to Lemma 5.4 (i), $y + \lambda_e(y)e \in O$ and so the set $\{t \in \mathbb{R}_+ : y + te \in O\}$ contains $\lambda_e(y)$. Hence we can define a function λ_{-1} from Y into $\mathbb{R}_+$ by the formula $\lambda_{-1}(y) := \inf\{t \in \mathbb{R}_+ : y + te \in O\}$ for all $y \in Y$. The properties of the function λ_{-1} are given in the following lemma.

LEMMA 5.7. *The following holds:*

(i) *for all $y \in Y$, $0 \leqslant \lambda_{-1}(y) \leqslant \lambda_e(y)$;*

(ii) *for all $y \in \operatorname{int} Y$, $\lambda_{-1}(y) < \lambda_e(y)$;*

(iii) *the function λ_{-1} is continuous.*

Proof.

(i) It is clear.

(ii) Let $y \in \operatorname{int} Y$. Clearly we have $\lambda_e(y) > 0$. Moreover, recalling that $y + \lambda_e(y)e \in O$ and that that O is an open set for the induced topology on Y, there exists $0 < t < \lambda_e(y)$ such that $y - te \in O$. Following the definition of λ_{-1}, we have $\lambda_{-1}(y) \leqslant t$, and so the inequality is verified.

(iii) Let $\{y_k\}$ be a sequence in Y converging to an element y of Y.

Firstly let us prove that the sequence $\{\lambda_{-1}(y_k)\}$ is bounded. From (i), it suffices to prove that the sequence $\{\lambda_e(y_k)\}$ is bounded. Since the sequence $\{y_k\}$ converges, there exists $z \in Y$ such that $y_k \geqslant z$ for all integers k. Consequently $y_k + \lambda_e(y_k)e \geqslant z + \lambda_e(y_k)e$, which implies that $z + \lambda_e(y_k)e \in Y$ from the free disposal assumption. By definition of $\lambda_e(z)$, we obtain $\lambda_e(y_k) \leqslant \lambda_e(z)$. Hence the sequence $\{\lambda_{-1}(y_k)\}$ is bounded.

It then suffices to prove that $\lambda_{-1}(y)$ is the unique point of accumulation of this sequence. Let $\{\lambda_{-1}(y_{k_j})\}_j$ be a subsequence which converges to a nonnegative number t. Let us show that $t = \lambda_{-1}(y)$.

Firstly we show the inequality $t \leqslant \lambda_{-1}(y)$. It is true for $t = 0$. If $t > 0$, for j large enough $\lambda_{-1}(y_{k_j}) > 0$, and following the definition of $\lambda_{-1}(y_{k_j})$ we get

$y_{k_j} + (1 - 1/(j+1))\lambda_{-1}(y_{k_j})e \in Y \setminus O$. Letting $j \to +\infty$ yields $y + te \in Y \setminus O$, recalling that $Y \setminus O$ is closed. Hence, by definition of $\lambda_{-1}(y)$, we deduce that $t \leqslant \lambda_{-1}(y)$.

Suppose now that the inequality is strict, and set $\epsilon := (\lambda_{-1}(y) - t)/2 > 0$. We have $0 \leqslant t + \epsilon < \lambda_{-1}(y)$. Then, by definition of $\lambda_{-1}(y)$, $y + (t + \epsilon)e \in Y \setminus O$. Since $y + (t + \epsilon)e \gg y_{k_j} + (\lambda_{-1}(y_{k_j}) + \epsilon/2)e$ for j large enough, we deduce $y_{k_j} + (\lambda_{-1}(y_{k_j}) + \epsilon/2)e \in Y \setminus O$. Hence, by definition of $\lambda_{-1}(y_{k_j})$, we conclude that $\lambda_{-1}(y_{k_j}) + \epsilon/2 \leqslant \lambda_{-1}(y_{k_j})$, which is impossible. Thus the equality $t = \lambda_{-1}(y)$ is proven. □

We can now define another function r_{-1} from A into Y such that for all $a \in A$ $r_{-1}(a) := a + (\lambda_{-1}(a) + \lambda_e(a))/2\, e$. The following properties of the function r_{-1} are straightforward consequences of Lemmas 5.6 and 5.7.

LEMMA 5.8. *The following holds:*

(i) *for all* $a \in A$, $r_{-1}(a) \geqslant a$;

(ii) *for all* $b \in B$, $r_{-1}(b) = b$;

(iii) $r_{-1}(A) \subset O \cap A$;

(iv) *the function* r_{-1} *is continuous.*

In this context, it is easy to prove that the function $\tilde{f}_k := \tilde{r}_k \circ \cdots \circ \tilde{r}_0 \circ r_{-1}$ from A into Y is continuous for each integer k, and to prove that the sequence $\{\tilde{f}_k\}$ converges pointwise to a function $\tilde{f} : A \to Y$ which is a retraction from A into B. □

6. Conclusion

We have shown that the efficient frontier of any nonempty simply shaded set with compact sections in finite dimensional Euclidean space ordered by the positive cone is strongly contractible. By similar technics, it is not difficult to extend this result to finite dimensional Euclidean space ordered by a polyhedral cone with a nonempty interior, or more generally by a closed convex cone with a nonempty interior and with a denumerable number of extremal rays (see [2] for the definition of simply shaded set in this context). We conjecture that, under assumptions to be defined, it is also valid in a Banach space ordered by a closed convex cone with a nonempty interior and with a bounded base (see Luc [6, Theorem 3.6, p. 143] for the convex case).

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